MOD 2 COHOMOLOGY OF 2-LOCAL FINITE GROUPS OF LOW RANK

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A . We determine the mod 2 cohomology over the Steenrod algebra $\mathcal{A}_2$ of the classifying space of a free loop group LG for $G = Spin(7)$, $Spin(8)$, $Spin(9)$, F_4, and $DI(4)$. Then we show that it is isomorphic as algebras over $\mathcal{A}_2$ to the mod 2 cohomology of the classifying space of a certain 2-local finite group of type G.

1. I

In [Ku], Kuribayashi considered the cohomology of the free loop space LX over a space X by developing a tool called the *module derivation*, which is a map from the cohomology of X to that of LX with degree -1 having some nice properties.

A free loop space LX can be considered as the homotopy fixed points space of the identity map of X. In addition, for a prime p, the mod p homotopy type of the classifying space of a certain finite group also occurs as the Bousfield and Kan p-completion of the homotopy fixed points space of the self-map of the classifying space BG of a compact Lie group G, namely the unstable Adams operation ([F]). From this point of view, Kishimoto and Kono ([KK]) generalized Kuribayashi's method to calculate the cohomology of the homotopy fixed points space of a self-map ϕ of X, which they call the twisted loop space $\mathbb{L}_\phi X$.

When the cohomology of BG is polynomial algebra, the calculation of the cohomology of the homotopy fixed points space sometimes reduces to an easy computation using their method. In this note, we give actual computations for the mod 2 cohomology over the Steenrod algebra $\mathcal{A}_2$ of the classifying spaces of free loop groups LG of G and 2-local finite groups ([BM]) of type G with $G = Spin(7), Spin(8), Spin(9), F_4$, and $DI(4)$ the finite loop space at prime 2 constructed by Dwyer and Wilkerson ([DW]). And we have the following Theorem:

Theorem 1.1. *We have the following isomorphisms of algebras over the Steenrod algebra $\mathcal{A}_2$.*

$$H^*(BLSpin(n); \mathbb{Z}/2) \cong H^*(Spin_n(q); \mathbb{Z}/2) \quad (n = 7, 8, 9)$$
$$H^*(BLF_4; \mathbb{Z}/2) \cong H^*(F_4(q); \mathbb{Z}/2)$$
$$H^*(BLDI(4); \mathbb{Z}/2) \cong H^*(BSol(q); \mathbb{Z}/2),$$

where q is an odd prime power.

Note that $H^*(BLG; \mathbb{Z}/2) = H^*(LBG; \mathbb{Z}/2)$ (see for example [Ku, §2]). For $G = Spin(7)$, $Spin(8)$, $Spin(9)$, F_4, and $DI(4)$, the explicit computations for $H^*(LBG; \mathbb{Z}/2)$ are given in the sections §3, §4 and §5.

Acknowledgment. We would like to thank Prof. Akira Kono for various suggestions.

2000 *Mathematics Subject Classification.* Primary 55R35; Secondary 55S10.

Key words and phrases. mod 2 cohomology, free loop groups, 2-local finite groups .

The first author is partially supported by the Grant-in-Aid for JSPS Fellows 182641.

2. M

Here we summarize the result of [KK] necessary for our purpose. Let ϕ be a based self-map of a based space X. The twisted loop space $\mathbb{L}_\phi X$ of X is defined in the following pull-back diagram:

$$\begin{tikzcd} \mathbb{L}_\phi X \arrow[r] \arrow[d] & X^{[0,1]} \arrow[d, "e_0\times e_1"] \\ X \arrow[r, "1\times\phi"] & X\times X \end{tikzcd}$$

where e_i $(i = 0, 1)$ is the evaluation at i. In other words, $\mathbb{L}_\phi X$ is a space of all continuous maps l from the interval $[0, 1]$ to X which satisfy $l(0) = \phi(l(1))$.

The twisted tube $\mathbb{T}_\phi X$ of X is defined by

$$\mathbb{T}_\phi X = \frac{[0,1]\times X}{(0,x)\simeq(1,\phi(x))}$$

and there is a canonical inclusion $\iota : X \hookrightarrow \mathbb{T}_\phi X$, where $\iota(x) = (0, x)$.

Remark 1. When ϕ is the identity map, then $\mathbb{L}_\phi X$ is merely the free loop space LX and $\mathbb{T}_\phi X = S^1 \times X$.

The cohomology of $\mathbb{T}_\phi X$ and X is related by the Wang exact sequence

$$\text{(A)} \qquad \cdots H^{n-1}(X;R) \xrightarrow{1-\phi^*} H^{n-1}(X;R) \xrightarrow{\delta} H^n(\mathbb{T}_\phi X;R) \xrightarrow{\iota^*} H^n(X;R) \xrightarrow{1-\phi^*} H^n(X;R)\cdots,$$

where R is any commutative ring. In particular, this exact sequence splits off to the short exact sequences when $H^*(\phi; R)$ is the identity map.

Let ev be the evaluation map

$$\begin{array}{rcl} S^1\times LX & \to & X \\ (t,l) & \mapsto & l(t). \end{array}$$

Then for any commutative ring R, a map $\sigma_X : H^*(X; R) \to H^{*-1}(LX; R)$ is defined by the following equation:

$$ev^*(x) = s\otimes\sigma_X(x) + 1\otimes x, \quad (x\in H^*(X;R)),$$

where $s \in H^1(S^1; R)$ is a generator. On the other hand, we define a map $in : \mathbb{L}_\phi X \to L\mathbb{T}_\phi X$ by

$$\begin{array}{rcl} \mathbb{L}_\phi X & \to & L\mathbb{T}_\phi X \\ l & \mapsto & t\mapsto (t,l(t)). \end{array}$$

Then the twisted cohomology suspension is defined by the following composition

$$\hat{\sigma}_\phi : H^*(\mathbb{T}_\phi X;R) \xrightarrow{\sigma_{\mathbb{T}_\phi X}} H^{*-1}(L\mathbb{T}_\phi X;R)) \xrightarrow{in^*} H^{*-1}(\mathbb{L}_\phi X;R).$$

Moreover, if we have a section $r : H^*(X; R) \to H^*(\mathbb{T}_\phi X; R)$ of ι^*, we can define another map $\tilde{\sigma}_\phi = r\circ\hat{\sigma}_\phi : H^*(X; R) \to H^{*-1}(\mathbb{L}_\phi X; R)$.

Remark 2. When $\phi = Id$ the identity map, we can take the section $r = \pi^*$, where $\pi : \mathbb{T}_\phi X = S^1 \times X \to X$ is the projection. Then $\tilde{\sigma}_\phi = \sigma_X$ and $\tilde{\sigma}_\phi$ also coincides with Kuribayashi's module derivation $\mathcal{D}_X$ defined in [Ku].

The map $\tilde{\sigma}_\phi$ together with the Wang sequence above relates the cohomology of X to that of $\mathbb{L}_\phi X$.

We consider the following conditions:

$$(*)\begin{cases} \text{(i)} & H^*(X;\mathbb{Z}/2) \text{ is a polynomial algebra } \mathbb{Z}/2[x_1, x_2, \dots, x_l], \\ \text{(ii)} & H^*(\phi;\mathbb{Z}/2) \text{ is the identity map,} \\ \text{(iii)} & H^n(\phi;\mathbb{Z}/4) \text{ is the identity map for all odd } n \text{ and } n \equiv 0 \mod 4. \end{cases}$$

Then the result of [KK] specializes to the following Proposition.

Proposition 2.1 (Kishimoto-Kono)**.** *Assume that (i) and (ii) in the conditions (*) are satisfied. Suppose that there is a section $r : H^*(X;\mathbb{Z}/2) \to H^*(\mathbb{T}_\phi X;\mathbb{Z}/2)$ of ι^*, which commutes with the Steenrod operations. Then we have*

1. *$H^*(\mathbb{L}_\phi X;\mathbb{Z}/2) = \mathbb{Z}/2[e^*(x_1), e^*(x_2), \dots, e^*(x_l)] \otimes \Delta(\tilde{\sigma}_\phi(x_1), \tilde{\sigma}_\phi(x_2), \dots, \tilde{\sigma}_\phi(x_l))$, where $e : \mathbb{L}_\phi X \to X$ is the evaluation at 0.*
2. *$\tilde{\sigma}_\phi(xy) = \tilde{\sigma}_\phi(x)e^*(y) + e^*(x)\tilde{\sigma}_\phi(y)$ for $x, y \in H^*(X;\mathbb{Z}/2)$*
3. *$\tilde{\sigma}_\phi$ commutes with the Steenrod operations.*

In the rest of this note, we restrict ourselves to the case when $X = BG$, where G is either $Spin(7)$, $Spin(8)$, $Spin(9)$, F_4, or $DI(4)$.

When ϕ is the identity map, $\mathbb{L}_\phi BG$ is merely the free loop space LBG, which is homotopy equivalent to BLG (see for example [Ku, §2]). Then the conditions (*) is trivially satisfied. Moreover, we can take π^* as a section r of ι^* which commutes with the Steenrod operations, where $\pi : S^1 \times X \to X$ is the projection. Hence we can calculate $H^*(BLG;\mathbb{Z}/2)$ by above Proposition.

For $G = Spin(7), Spin(8), Spin(9), F_4$ and a odd prime power q, there is a self-map ψ^q of BG called the unstable Adams operation of degree q ([W]), where $H^{2r}(\psi^q;\mathbb{Q})$ is multiplication by q^r. When $\phi = \psi^q$, the conditions (*) is satisfied. The Bousfield and Kan 2-completion ([BK]) of $\mathbb{L}_\phi BG$ is known to be homotopy equivalent to that of the classifying space of a finite Chevalley group of type $G(q)$ ([F]). Hence we can calculate $H^*(G(q);\mathbb{Z}/2)$ by above Proposition.

For $G = DI(4)$ and a odd prime power q, Notbohm ([N]) showed that there is a self-map ψ^q of $BDI(4)$ also called the unstable Adams operation of degree q ([N]), where $H^{2r}(\psi^q;\mathbb{Q}_2^\wedge)$ is multiplication by q^r. When $\phi = \psi^q$, the conditions (*) is satisfied. Using this map, Benson ([B]) defined the classifying space $BSol(q)$ of an exotic 2-local finite group as $L_{\psi^q}BDI(4)$ which can be regarded as the "classifying space" of Solomon's non-existent finite group ([S]). Hence we can calculate $H^*(BSol(q);\mathbb{Z}/2)$ by above Proposition.

To sum up, Theorem 1.1 reduces to the computation of $H^*(\mathbb{L}_\phi BG;\mathbb{Z}/2)$, where ϕ is the identity map or ψ^q. In the following sections, our main observation is to construct a section $r : H^*(BG;\mathbb{Z}/2) \to H^*(\mathbb{T}_\phi BG;\mathbb{Z}/2)$ which commutes with Steenrod operations when $\phi = \psi^q$ to show the following:

Theorem 2.1. *Let $G = Spin(7), Spin(8), Spin(9), F_4$ or $DI(4)$. Then $H^*(LBG;\mathbb{Z}/2) \simeq H^*(\mathbb{L}_{\psi^q}BG;\mathbb{Z}/2)$ as algebras over the Steenrod algebra $\mathcal{A}_2$, where ψ^q is the unstable Adams operation of degree an odd prime power q.*

3. C $G = Spin(7), Spin(8), Spin(9)$

The mod 2 cohomology over $\mathcal{A}_2$ of $BSpin(7), BSpin(8)$ and $BSpin(9)$ are well known ([Q, Ko]). Since we rely on these results, we recall them here.

$H^*(BSpin(7);\mathbb{Z}/2) = \mathbb{Z}[w_4, w_6, w_7, w_8]$ and the action of $\mathcal{A}_2$ is determined by:

$$\begin{array}{ccccc} & w_4 & w_6 & w_7 & w_8 \\ Sq^1 & 0 & w_7 & 0 & 0 \\ Sq^2 & w_6 & 0 & 0 & 0 \\ Sq^4 & w_4^2 & w_4w_6 & w_4w_7 & w_4w_8. \end{array}$$

$H^*(BSpin(8);\mathbb{Z}/2) = \mathbb{Z}[w_4, w_6, w_7, w_8, e_8]$ and the action of $\mathcal{A}_2$ is determined by:

$$\begin{array}{cccccc} & w_4 & w_6 & w_7 & w_8 & e_8 \\ Sq^1 & 0 & w_7 & 0 & 0 & 0 \\ Sq^2 & w_6 & 0 & 0 & 0 & 0 \\ Sq^4 & w_4^2 & w_4w_6 & w_4w_7 & w_4w_8 & w_4e_8. \end{array}$$

$H^*(BSpin(9);\mathbb{Z}/2) = \mathbb{Z}[w_4, w_6, w_7, w_8, e_{16}]$ and the action of $\mathcal{A}_2$ is determined by:

$$\begin{array}{cccccc} & w_4 & w_6 & w_7 & w_8 & e_{16} \\ Sq^1 & 0 & w_7 & 0 & 0 & 0 \\ Sq^2 & w_6 & 0 & 0 & 0 & 0 \\ Sq^4 & w_4^2 & w_4w_6 & w_4w_7 & w_4w_8 & 0 \\ Sq^8 & 0 & 0 & 0 & w_8^2 & w_8e_{16} + w_4^2e_{16}. \end{array}$$

Based on these results, we compute the mod 2 cohomology of classifying spaces of free loop groups LG for $G = Spin(7), Spin(8), Spin(9)$.

Proposition 3.1. *$H^*(LBSpin(7);\mathbb{Z}/2) = \mathbb{Z}/2[v_4, v_6, v_7, v_8, y_3, y_5, y_7]/I$ $(|v_i| = i, |y_i| = i)$, where I is the ideal generated by*

$$\{y_5^2 + y_3^2v_4 + y_3v_7, y_3^4 + y_3^2v_6 + y_5v_7, y_7^2 + y_3^2v_8 + y_7v_7\}.$$

The action of $\mathcal{A}_2$ is determined by:

$$\begin{array}{cccccccc} & v_4 & v_6 & v_7 & v_8 & y_3 & y_5 & y_7 \\ Sq^1 & 0 & v_7 & 0 & 0 & 0 & y_3^2 & 0 \\ Sq^2 & v_6 & 0 & 0 & 0 & y_5 & 0 & 0 \\ Sq^4 & v_4^2 & v_4v_6 & v_4v_7 & v_4v_8 & 0 & y_3v_6 + y_5v_4 & y_3v_8 + y_7v_4. \end{array}$$

Proof. We apply Proposition 2.1 when $\phi = Id$ the identity map. Since $\mathbb{L}_{Id}BG = LBG$ and $\mathbb{T}_\phi X = S^1 \times X$, we can use Proposition 2.1 with $r = \pi^*$, where $\pi : S^1 \times X \to X$ is the projection.

We take $v_i = e^*(w_i),\ y_{i-1} = \tilde{\sigma}_{Id}(w_i)\quad (i = 4, 6, 7, 8)$. Then by Proposition 2.1 (1), we have $H^*(LBSpin(7);\mathbb{Z}/2) = \mathbb{Z}/2[v_4, v_6, v_7, v_8] \otimes \Delta[y_3, y_5, y_6, y_7]$. By Proposition 2.1 (2) and (3), the action of $\mathcal{A}_2$ on the generator v_i $(i = 4, 6, 7, 8)$ is obvious, and on y_i $(i = 3, 5, 7)$ we calculate as follows:

$$\begin{aligned}
&Sq^1y_3 = Sq^1\tilde{\sigma}_{Id}(w_4) = \tilde{\sigma}_{Id}(Sq^1w_4) = 0\\
&Sq^2y_3 = \tilde{\sigma}_{Id}(Sq^2w_4) = \tilde{\sigma}_{Id}(w_6) = y_5\\
&Sq^1y_5 = \tilde{\sigma}_{Id}(Sq^1w_6) = y_6\\
&Sq^2y_5 = \tilde{\sigma}_{Id}(Sq^2w_6) = 0\\
&Sq^4y_5 = \tilde{\sigma}_{Id}(Sq^4w_6) = \tilde{\sigma}_{Id}(w_4w_6) = \tilde{\sigma}_{Id}(w_4)e^*(w_6) + \tilde{\sigma}_{Id}(w_6)e^*(w_4) = y_3v_6 + y_5v_4\\
&Sq^1y_7 = \tilde{\sigma}_{Id}(Sq^1w_8) = 0\\
&Sq^2y_7 = \tilde{\sigma}_{Id}(Sq^2w_8) = 0\\
&Sq^4y_7 = \tilde{\sigma}_{Id}(Sq^4w_8) = \tilde{\sigma}_{Id}(w_4w_8) = y_3v_8 + y_7v_4.
\end{aligned}$$

On the other hand, with the aid of the Adem relations, we can determine the ring structure as follows:

$$
\begin{aligned}
y_3^2 &= Sq^3 y_3 = Sq^1 Sq^2 y_3 = Sq^1 y_5 = y_6 \\
y_5^2 &= Sq^5 y_5 = Sq^1 Sq^4 y_5 = Sq^1(y_5 v_4 + y_3 v_6) = y_6 v_4 + y_3 v_7 = y_3^2 v_4 + y_3 v_7 \\
y_7^2 &= Sq^7 y_7 = Sq^1 Sq^2 Sq^4 y_7 = Sq^1 Sq^2 (y_3 v_8 + y_7 v_4) = Sq^1(y_5 v_8 + y_7 v_6) = y_3^2 v_8 + y_7 v_7 \\
y_3^4 &= y_6^2 = Sq^6 y_6 = (Sq^2 Sq^4 + Sq^5 Sq^1) y_6 = \tilde{\sigma}_\phi((Sq^2 Sq^4 + Sq^5 Sq^1) w_7) \\
&= \tilde{\sigma}_\phi(Sq^2 w_4 w_7) = \tilde{\sigma}_\phi(w_6 w_7) = y_5 v_7 + v_6 y_3^2.
\end{aligned}
$$

□

Now we proceed to the computation of the mod 2 cohomology algebra over $\mathcal{A}_2$ of a finite Chevalley group of type $Spin_7(q)$ for an odd prime power q. The Bousfield-Kan 2-completion of the classifying space of $Spin_7(q)$ is shown to be homotopy equivalent to the Bousfield-Kan 2-completion of $\mathbb{L}_{\psi^q} BSpin(7)$ by Friedlander ([F]) where ψ^q is the unstable Adams operation of degree q. Thus $H^*(\mathbb{L}_{\psi^q} BSpin(7); \mathbb{Z}/2) \cong H^*(Spin_7(q); \mathbb{Z}/2)$ as algebras over the Steenrod algebra $\mathcal{A}_2$.

Proposition 3.2. *For $\phi = \psi^q$ the unstable Adams operation of degree an odd prime power q, $H^*(\mathbb{L}_\phi BSpin(7); \mathbb{Z}/2)$ is isomorphic to $H^*(LBSpin(7); \mathbb{Z}/2)$ as algebras over $\mathcal{A}_2$.*

Proof. By Proposition 2.1, we only have to construct a section r of $\iota^* : H^*(\mathbb{T}_\phi BSpin(7); \mathbb{Z}/2) \to H^*(BSpin(7); \mathbb{Z}/2)$ which commutes with the Steenrod operations. To do so, we carefully choose an element of $(\iota^*)^{-1}(w_i) \subset H^i(\mathbb{T}_\phi BSpin(7); \mathbb{Z}/2)$ for each generator w_i of $H^i(BSpin(7); \mathbb{Z}/2)$ so that the action of $\mathcal{A}_2$ on it is compatible with that on w_i.

As mentioned in the first section, now the conditions (*) are satisfied and the Wang sequences (A) for $R = \mathbb{Z}/2$ and $\mathbb{Z}/4$ split to the short exact sequences

$$0 \to H^{*-1}(BSpin(7); \mathbb{Z}/2) \xrightarrow{\delta} H^*(\mathbb{T}_\phi BSpin(7); \mathbb{Z}/2) \xrightarrow{\iota^*} H^*(BSpin(7); \mathbb{Z}/2) \to 0,$$

$$0 \to H^{n-1}(BSpin(7); \mathbb{Z}/4) \xrightarrow{\delta} H^n(\mathbb{T}_\phi BSpin(7); \mathbb{Z}/4) \xrightarrow{\iota^*} H^{n-1}(BSpin(7); \mathbb{Z}/4) \to 0,$$

where $n \equiv 0 \mod 4$.

Let $u_4 \in H^4(\mathbb{T}_\phi BSpin(7); \mathbb{Z}/2) \simeq \mathbb{Z}/2$ be a generator and we put $u_6 = Sq^2 u_4$, $u_7 = Sq^1 u_6$. By the Wang sequence for $\mathbb{Z}/2$, we have that $H^8(\mathbb{T}_\phi BSpin(7); \mathbb{Z}/2)$ is generated by $\delta(w_7)$ and any element $u_8' \in (\iota^*)^{-1}(w_8)$. Note that w_8 is the mod 2 reduction of a generator of $H^8(BSpin(7); \mathbb{Z})$. Thus from the Wang sequence for $\mathbb{Z}/4$, we have that u_8' is in the image under ρ, where ρ is the mod 2 reduction map in the following Bockstein sequence

$$\to H^8(\mathbb{T}_\phi BSpin(7); \mathbb{Z}/2) \to H^8(\mathbb{T}_\phi BSpin(7); \mathbb{Z}/4) \xrightarrow{\rho} H^8(\mathbb{T}_\phi BSpin(7); \mathbb{Z}/2)$$
$$\xrightarrow{Sq^1} H^9(\mathbb{T}_\phi BSpin(7); \mathbb{Z}/2) \to \cdots .$$

Therefore we have $Sq^1(u_8') = 0$. Furthermore, using the Wang sequence for $\mathbb{Z}/2$, we have $Sq^2 u_8' = 0$ since $H^9(BSpin(7); \mathbb{Z}/2) = 0$ and $\iota^*(Sq^2 u_8') = Sq^2 \iota^*(u_8') = Sq^2 w_8 = 0$.

Now we want to replace u_8' with the one compatible with the action of Sq^4 on w_8, without changing the action of Sq^i for $i < 8$.

Since $H^{11}(BSpin(7); \mathbb{Z}/2)) \cong \mathbb{Z}/2$ is generated by $w_4 w_7$ and $\iota^*(Sq^4 u_8') = Sq^4 w_8 = w_4 w_8$, we have $Sq^4 u_8' = u_4 u_8' + \epsilon\delta(w_4 w_7)$, where $\epsilon = 0$ or 1. We put $u_8 = u_8' + \epsilon\delta(w_7)$. Since $\delta(w_4 w_7) = \delta(Sq^4 w_7) = Sq^4 \delta(w_7)$, we have $Sq^4(u_8) = u_4 u_8$. Since $Sq^i \delta(w_7) = \delta(Sq^i w_7) = 0$, $(i = 1, 2)$, we have $Sq^i(u_8) = 0$, $(i = 1, 2)$.

Take r to be the ring homomorphism defined by $r(w_i) = u_i$ ($i = 4, 6, 7, 8$), then r is a section of ι^* which commutes with the Steenrod algebra $\mathcal{A}_2$. □

Now we proceed to the case when $G = Spin(8)$.

Proposition 3.3. *$H^*(LBSpin(8); \mathbb{Z}/2) = \mathbb{Z}/2[v_4, v_6, v_7, v_8, f_8, y_3, y_5, y_7, z_7]/I$ ($|v_i| = i, |y_i| = i, |v_8| = 8, |z_7| = 7$), where I is the ideal generated by*

$$\{y_5^2 + y_3^2 v_4 + y_3 v_7, y_3^4 + y_3^2 v_6 + y_5 v_7, y_7^2 + y_3^2 v_8 + y_7 v_7, z_7^2 + y_3^2 f_8 + z_7 v_7\}.$$

The action of $\mathcal{A}_2$ is determined by:

$$\begin{array}{cccccccccc}
 & v_4 & v_6 & v_7 & v_8 & f_8 & y_3 & y_5 & y_7 & z_7 \\
Sq^1 & 0 & v_7 & 0 & 0 & 0 & 0 & y_3^2 & 0 & 0 \\
Sq^2 & v_6 & 0 & 0 & 0 & 0 & y_5 & 0 & 0 & 0 \\
Sq^4 & v_4^2 & v_4 v_6 & v_4 v_7 & v_4 v_8 & v_4 f_8 & 0 & y_3 v_6 + y_5 v_4 & y_3 v_8 + y_7 v_4 & y_3 f_8 + z_7 v_4.
\end{array}$$

Proof. Completely parallel to the case of $Spin(7)$ since the generator $e_8 \in H^8(BSpin(8); \mathbb{Z}/2)$ looks same as w_8. □

A finite Chevalley group of type $Spin_8(q)$ has the mod 2 cohomology algebra over $\mathcal{A}_2$ isomorphic to $H^*(\mathbb{L}_{\psi^q} BSpin(8); \mathbb{Z}/2)$, where q is an odd prime power. And we have

Proposition 3.4. *For $\phi = \psi^q$ the unstable Adams operation of degree an odd prime power q, $H^*(\mathbb{L}_\phi BSpin(8); \mathbb{Z}/2)$ is isomorphic to $H^*(LBSpin(8); \mathbb{Z}/2)$ as algebras over $\mathcal{A}_2$.*

Proof. We can construct a section r completely parallel to the case of $BSpin(7)$. □

Now we proceed to the case when $G = Spin(9)$.

Proposition 3.5. *$H^*(LBSpin(9); \mathbb{Z}/2) = \mathbb{Z}/2[v_4, v_6, v_7, v_8, f_{16}, y_3, y_5, y_7, z_{15}]/I$ ($|v_i| = i, |y_i| = i, |f_{16}| = 16, |z_{16}| = 16$), where I is the ideal generated by*

$$\{y_5^2 + y_3 v_7 + v_4 y_3^2, y_3^4 + y_3^2 v_6 + y_5 v_7, y_7^2 + y_3^2 v_8 + y_7 v_7, z_{15}^2 + v_7 v_8 z_{15} + v_7 y_7 f_{16} + y_3^2 v_8 f_{16}\}.$$

The action of $\mathcal{A}_2$ is determined by:

$$\begin{array}{cccccccccc}
 & v_4 & v_6 & v_7 & v_8 & f_{16} & y_3 & y_5 & y_7 & z_{15} \\
Sq^1 & 0 & v_7 & 0 & 0 & 0 & 0 & y_3^2 & 0 & 0 \\
Sq^2 & v_6 & 0 & 0 & 0 & 0 & y_5 & 0 & 0 & 0 \\
Sq^4 & v_4^2 & v_4 v_6 & v_4 v_7 & v_4 v_8 & 0 & 0 & y_3 v_6 + y_5 v_4 & y_3 v_8 + y_7 v_4 & 0 \\
Sq^8 & 0 & 0 & 0 & v_8^2 & v_8 f_{16} + v_4^2 f_{16} & 0 & 0 & 0 & J_1
\end{array}$$

where $J_1 = y_7 f_{16} + v_8 z_{15} + v_4^2 z_{15}$.

Proof. In dimensions lower than 9, the calculation is completely same as in the case of $BSpin(7)$. We take $f_{16} = e^*(e_{16})$, $z_{15} = \tilde{\sigma}_{Id}(e_{16})$. Then we have only to calculate the following:

$$\begin{aligned}
Sq^8 z_{15} &= \tilde{\sigma}_{Id}(Sq^8 e_{16}) = \tilde{\sigma}_{Id}(w_8 e_{16} + w_4^2 e_{16}) = y_7 f_{16} + v_8 z_{15} + v_4^2 z_{15} \\
z_{15}^2 &= Sq^{15} z_{15} = \tilde{\sigma}_{Id}(Sq^{15} e_{16}) = \tilde{\sigma}_{Id}(Sq^7 Sq^8 e_{16}) = \tilde{\sigma}_{Id}(Sq^7(w_8 e_{16} + w_4^2 e_{16})) \\
&= \tilde{\sigma}_{Id}(Sq^3 Sq^4(w_8) e_{16} + Sq^3 Sq^4(w_4^2) e_{16}) = \tilde{\sigma}_{Id}(Sq^1 Sq^2(w_4 w_8) e_{16} + Sq^3(w_6^2) e_{16}) \\
&= \tilde{\sigma}_{Id}(w_7 w_8 e_{16}) = v_7 v_8 \tilde{\sigma}_{Id}(e_{16}) + \tilde{\sigma}_{Id}(w_7 w_8) f_{16} = v_7 v_8 z_{15} + v_7 y_7 f_{16} + y_3^2 v_8 f_{16}.
\end{aligned}$$

□

A finite Chevalley group of type $Spin_9(q)$ has the mod 2 cohomology algebra over $\mathcal{A}_2$ isomorphic to $H^*(\mathbb{L}_\phi BSpin(9);\mathbb{Z}/2)$, where $\phi = \psi^q_{BSpin(9)}$ and q is an odd prime power. And we have

Proposition 3.6. *For $\phi = \psi^q_{BSpin(9)}$ the unstable Adams operation of degree an odd prime power q, $H^*(\mathbb{L}_\phi BSpin(9);\mathbb{Z}/2)$ is isomorphic to $H^*(LBSpin(9);\mathbb{Z}/2)$ as algebras over $\mathcal{A}_2$.*

Proof. In dimensions lower than 9, we can construct a section $r_{BSpin(9)}$ completely parallel to the case of $BSpin(7)$, namely $r_{BSpin(9)}(w_i) = u_i$ $(i = 4, 6, 7, 8)$.

Now the conditions (*) are satisfied and the Wang sequences (A) for $R = \mathbb{Z}/2$ and $\mathbb{Z}/4$ split to the short exact sequences

$$0 \to H^{*-1}(BSpin(9);\mathbb{Z}/2) \xrightarrow{\delta} H^*(\mathbb{T}_\phi BSpin(9);\mathbb{Z}/2) \xrightarrow{\iota^*} H^*(BSpin(9);\mathbb{Z}/2) \to 0,$$

$$0 \to H^{n-1}(BSpin(9);\mathbb{Z}/4) \xrightarrow{\delta} H^n(\mathbb{T}_\phi BSpin(9);\mathbb{Z}/4) \xrightarrow{\iota^*} H^{n-1}(BSpin(9);\mathbb{Z}/4) \to 0,$$

where $n \equiv 0 \mod 4$.

Using the Wang sequence for $\mathbb{Z}/4$ and the Bockstein sequence, we can choose an element $h'_{16} \in \ker(Sq^1) \subset (\iota^*)^{-1}(e_{16}) \subset H^{16}(\mathbb{T}_\phi BSpin(9);\mathbb{Z}/2)$ by the same observation for u_8 in the proof of $BSpin(7)$.

Then by the Wang sequence for $\mathbb{Z}/2$, we have $Sq^2 h'_{16} = \epsilon_1\delta(w_4w_6w_7)$ since $H^{17}(BSpin(9);\mathbb{Z}/2) \simeq \mathbb{Z}/2$ is generated by $w_4w_6w_7$ and $\iota^*(Sq^2h'_{16}) = Sq^2e_{16} = 0$. Then $Sq^2Sq^2h'_{16} = \epsilon_1\delta(Sq^2(w_4w_6w_7)) = \epsilon_1\delta(w_6^2w_7)$. Since $Sq^2Sq^2 = Sq^3Sq^1$ by Adem relation and $Sq^1h'_{16} = 0$, ϵ_1 must be 0.

Similarly we have $Sq^4h'_{16} = \epsilon_2\delta(w_4^3w_7) + \epsilon_3\delta(w_6^2w_7) + \epsilon_4\delta(w_4w_7w_8)$. Then we have $Sq^4Sq^4h'_{16} = \epsilon_2\delta(w_4w_6^2w_7) + \epsilon_3\delta(w_4w_6^2w_7) + \epsilon_4\delta(w_4^2w_7w_8)$. By Adem relation we have $Sq^4Sq^4h'_{16} = (Sq^7Sq^1 + Sq^6Sq^2)h'_{16} = 0$. Therefore we have $\epsilon_2 = \epsilon_3, \epsilon_4 = 0$. Put $h_{16} = h'_{16} + \epsilon_2\delta(w_4^2w_7)$, then we have $Sq^4h_{16} = 0$ since $Sq^4(w_4^2w_7) = w_4^3w_7 + w_6^2w_7$. Note that we also have $Sq^ih_{16} = 0$ $(i = 1, 2)$ since $Sq^i(w_4^2w_7) = 0$ $(i = 1, 2)$.

Similarly we have $Sq^8h_{16} = u_8h_{16} + u_4^2h_{16} + \epsilon_5\delta(w_4^4w_7) + \epsilon_6\delta(w_4^2w_7w_8) + \epsilon_7\delta(w_4w_6^2w_7) + \epsilon_8\delta(w_7w_8^2) + \epsilon_9\delta(w_7e_{16})$. By Adem relation $Sq^8Sq^8h_{16} = 0$ and we have $\epsilon_5 = \epsilon_7 = \epsilon_9 = 0, \epsilon_6 = \epsilon_8$. Replacing h_{16} by $h_{16} + \epsilon_6\delta(w_7w_8)$ we have $Sq^8h_{16} = u_8h_{16} + u_4^2h_{16}$ and $Sq^ih_{16} = 0$ $(i < 8)$.

Define $r_{BSpin(9)}(e_{16}) = h_{16}$, then $r_{BSpin(9)}$ is a section of ι^* which commutes with the Steenrod operations. ◻

4. C $G = F_4$

The same method applies for the case of simply connected, simple exceptional compact Lie group F_4.

We first recall the mod 2 cohomology of BF_4. Denote by i the classifying map of the canonical inclusion $Spin(9) \hookrightarrow F_4$. Kono determined the mod 2 cohomology algebra over $\mathcal{A}_2$ in [Ko] as follows:

$$H^*(BF_4;\mathbb{Z}/2) = \mathbb{Z}[x_4, x_6, x_7, x_{16}, x_{24}],$$

where $i^*(x_4) = w_4, i^*(x_6) = w_6, i^*(x_7) = w_7, i^*(x_{16}) = e_{16} + w_8^2, i^*(x_{24}) = w_8 e_{16}$ and the action of $\mathcal{A}_2$ is determined by:

$$
\begin{array}{cccccc}
 & x_4 & x_6 & x_7 & x_{16} & x_{24} \\
Sq^1 & 0 & x_7 & 0 & 0 & 0 \\
Sq^2 & x_6 & 0 & 0 & 0 & 0 \\
Sq^4 & x_4^2 & x_4x_6 & x_4x_7 & 0 & x_4x_{24} \\
Sq^8 & 0 & 0 & 0 & x_{24} + x_4^2x_{16} & x_4^2x_{24} \\
Sq^{16} & 0 & 0 & 0 & x_{16}^2 & x_{16}x_{24} + x_4x_6^2x_{24}.
\end{array}
$$

Then we can compute the mod 2 cohomology of classifying space of the free loop group LF_4 just as in the same manner in the previous section.

Proposition 4.1. *$H^*(LBF_4; \mathbb{Z}/2) = \mathbb{Z}/2[v_4, v_6, v_7, v_{16}, v_{24}, y_3, y_5, y_{15}, y_{23}]/I$ $(|v_i| = i, |y_i| = i)$, where I is the ideal generated by*

$$\{y_5^2 + y_3v_7 + v_4y_3^2, y_3^4 + v_6y_3^2 + y_5v_7, y_{15}^2 + v_7y_{23} + v_{24}y_3^2, y_{23}^2 + y_3^2v_{16}v_{24} + v_7v_{24}y_{15} + v_7v_{16}y_{23}\}.$$

The action of $\mathcal{A}_2$ is determined by:

$$
\begin{array}{cccccc}
 & v_4 & v_6 & v_7 & v_{16} & v_{24} \\
Sq^1 & 0 & v_7 & 0 & 0 & 0 \\
Sq^2 & v_6 & 0 & 0 & 0 & 0 \\
Sq^4 & v_4^2 & v_4v_6 & v_4v_7 & 0 & v_4v_{24} \\
Sq^8 & 0 & 0 & 0 & v_{24} + v_4^2v_{16} & v_4^2v_{24} \\
Sq^{16} & 0 & 0 & 0 & v_{16}^2 & v_{16}v_{24} + v_4v_6^2v_{24} \\
\hline
 & y_3 & y_5 & y_{15} & y_{23} & \\
Sq^1 & 0 & y_3^2 & 0 & 0 & \\
Sq^2 & y_5 & 0 & 0 & 0 & \\
Sq^4 & 0 & y_3v_6 + v_4y_5 & 0 & y_3v_{24} + v_4y_{23} & \\
Sq^8 & 0 & 0 & y_{23} + v_4^2y_{15} & v_4^2y_{23} & \\
Sq^{16} & 0 & 0 & 0 & J_2 &
\end{array}
$$

where $J_2 = v_{24}y_{15} + v_{16}y_{23} + y_3v_6^2v_{24} + v_4v_6^2y_{23}$.

Proof. We take $v_i = e^*(x_i)$, $y_{i-1} = \tilde{\sigma}_{Id}(x_i)$ $(i = 4, 6, 7, 16, 24)$. Then In dimensions lower than 9, the calculation is completely parallel to the the case of $BSpin(9)$. And the rest are

as follows:

$$
\begin{aligned}
Sq^1 y_{15} &= \tilde{\sigma}_{Id}(Sq^1 x_{16}) = 0\\
Sq^2 y_{15} &= \tilde{\sigma}_{Id}(Sq^2 x_{16}) = 0\\
Sq^4 y_{15} &= \tilde{\sigma}_{Id}(Sq^4 x_{16}) = 0\\
Sq^8 y_{15} &= \tilde{\sigma}_{Id}(Sq^8 x_{16}) = \tilde{\sigma}_{Id}(x_{24} + x_4^2 x_{16}) = y_{23} + v_4^2 y_{15}\\
Sq^1 y_{23} &= \tilde{\sigma}_{Id}(Sq^1 x_{24}) = 0\\
Sq^2 y_{23} &= \tilde{\sigma}_{Id}(Sq^2 x_{24}) = 0\\
Sq^4 y_{23} &= \tilde{\sigma}_{Id}(Sq^4 x_{24}) = \tilde{\sigma}_{Id}(x_4 x_{24}) = y_3 v_{24} + v_4 y_{23}\\
Sq^8 y_{23} &= \tilde{\sigma}_{Id}(Sq^8 x_{24}) = \tilde{\sigma}_{Id}(x_4^2 x_{24}) = v_4^2 y_{23}\\
Sq^{16} y_{23} &= \tilde{\sigma}_{Id}(Sq^{16} x_{24}) = \tilde{\sigma}_{Id}(x_{16} x_{24} + x_4 x_6^2 x_{24}) = y_{15} v_{24} + v_{16} y_{23} + y_3 v_6^2 v_{24} + v_4 v_6^2 y_{23}\\
y_{15}^2 &= Sq^{15} y_{15} = \tilde{\sigma}_{Id}(Sq^{15} x_{16}) = \tilde{\sigma}_{Id}(Sq^7 Sq^8 x_{16}) = \tilde{\sigma}_{Id}(Sq^7 x_{24}) = \tilde{\sigma}_{Id}(x_7 x_{24}) = v_7 y_{23} + y_3^2 v_{24}\\
y_{23}^2 &= Sq^{23} y_{23} = \tilde{\sigma}_{Id}(Sq^{23} x_{24}) = \tilde{\sigma}_{Id}(Sq^7 Sq^{16} x_{24}) = \tilde{\sigma}_{Id}(Sq^7(x_{16} x_{24} + x_4 x_6^2 x_{24}))\\
&\quad \tilde{\sigma}_{Id}(x_{16} Sq^7(x_{24}) + Sq^7 Sq^4(x_6^2 x_{24}))) = \tilde{\sigma}_{Id}(x_7 x_{16} x_{24}) = y_3^2 v_{16} v_{24} + v_7 y_{15} v_{24} + v_7 v_{16} y_{23}.
\end{aligned}
$$

□

A finite Chevalley group of type $F_4(q)$ has the mod 2 cohomology algebra over $\mathcal{A}_2$ isomorphic to $H^*(\mathbb{L}_\phi BF_4; \mathbb{Z}/2)$, where $\phi = \psi^q_{BF_4}$ and q is an odd prime power. And we have

Proposition 4.2. *For $\phi = \psi^q_{BF_4}$ the unstable Adams operation of degree an odd prime power q, $H^*(\mathbb{L}_\phi BF_4; \mathbb{Z}/2)$ is isomorphic to $H^*(LBF_4; \mathbb{Z}/2)$ as algebras over $\mathcal{A}_2$.*

Proof. By [JMO] the following diagram is homotopy commutative

$$
\begin{tikzcd}
BSpin(9)^\wedge_2 \arrow[r, "\psi^q_{BSpin(9)}"] \arrow[d, "(i)^\wedge_2"] & BSpin(9)^\wedge_2 \arrow[d, "(i)^\wedge_2"] \\
(BF_4)^\wedge_2 \arrow[r, "\psi^q_{BF_4}"] & (BF_4)^\wedge_2
\end{tikzcd},
$$

where $X^\wedge_2$ is the Bousfield-Kan 2-completion of X. By the naturality of the construction of the twisted tube, there is a map $\mathbb{T}_\phi(i)^\wedge_2 : \mathbb{T}_\phi BSpin(9)^\wedge_2 \to \mathbb{T}_\phi(BF_4)^\wedge_2$, which makes the following diagram commute:

$$
\begin{tikzcd}
0 \arrow[r] & H^{*-1}(BSpin(9); \mathbb{Z}/2) \arrow[r] & H^*(\mathbb{T}_\phi BSpin(9); \mathbb{Z}/2) \arrow[r, "\iota^*_{BSpin(9)}"] & H^*(BSpin(9); \mathbb{Z}/2) \arrow[r] & 0 \\
0 \arrow[r] & H^{*-1}(BF_4; \mathbb{Z}/2) \arrow[r] \arrow[u, "i^*"] & H^*(\mathbb{T}_\phi BF_4; \mathbb{Z}/2) \arrow[r, "\iota^*_{BF_4}"] \arrow[u, "(\mathbb{T}_\phi(i)^\wedge_2)^*"] & H^*(BF_4; \mathbb{Z}/2) \arrow[r] \arrow[u, "i^*"] & 0,
\end{tikzcd}
$$

where the horizontal lines are the Wang sequences. Since i^* is injective for degrees less than 48, so is $(\mathbb{T}_\phi(i)^\wedge_2)^*$. Then we can define a section r_{BF_4} of $\iota^*_{BF_4}$ as

$$
r_{BF_4}(x_i) = ((\mathbb{T}_\phi(i)^\wedge_2)^*)^{-1} \circ r_{BSpin(9)} \circ i^*(x_i), \quad (i = 4, 6, 7, 16, 24),
$$

where $r_{BSpin(9)}$ is the section of $\iota^*_{BSpin(9)}$ constructed in the previous section. For the commutativity with the Steenrod operations, we have only to consider the degrees less than or equal to $|Sq^{16}x_{24}| = 40$, thus we have that r commutes with the Steenrod algebra. □

5. C $G = DI(4)$

In [DW], Dwyer and Wilkerson constructed a finite loop space $DI(4)$, whose classifying space $BDI(4)$ has the mod 2 cohomology isomorphic to the mod 2 Dickson invariant of rank 4, that is, $H^*(BDI(4);\mathbb{Z}/2) = \mathbb{Z}/2[x_8, x_{12}, x_{14}, x_{15}]$, where $|x_j| = j$. The action of $\mathcal{A}_2$ is determined by:

	x_8	x_{12}	x_{14}	x_{15}
Sq^1	0	0	x_{15}	0
Sq^2	0	x_{14}	0	0
Sq^4	x_{12}	0	0	0
Sq^8	x_8^2	x_8x_{12}	x_8x_{14}	x_8x_{15}.

In ([B]), Benson defined the classifying space $BSol(q)$ of an exotic 2-local finite group as $L_{\psi^q}BDI(4)$, where q is an odd prime power and ψ^q is the unstable Adams operation of degree q constructed in [N]. Recently in [G], Grbić calculated the mod 2 cohomology of $BSol(q)$ over $\mathcal{A}_2$ by using the Eilenberg-Moore spectral sequence. Here we confirm it by the same method in previous sections. To do so, we first calculate the mod 2 cohomology of the free loop space $LBDI(4)$.

Proposition 5.1. *$H^*(LBDI(4);\mathbb{Z}/2) = \mathbb{Z}/2[v_8, v_{12}, v_{14}, v_{15}, y_7, y_{11}, y_{13}]/I$ ($|v_i| = i, |y_i| = i$), where I is the ideal generated by*

$$\{y_7^4 + y_{13}v_{15} + v_{14}y_7^2,\ y_{11}^2 + v_7y_{15} + v_8y_7^2,\ y_{13}^2 + y_{11}v_{15} + v_{12}y_7^2\}.$$

The action of $\mathcal{A}_2$ is determined by:

	v_8	v_{12}	v_{14}	v_{15}	y_7	y_{11}	y_{13}
Sq^1	0	0	v_{15}	0	0	0	y_7^2
Sq^2	0	v_{14}	0	0	0	y_{13}	0
Sq^4	v_{12}	0	0	0	y_{11}	0	0
Sq^8	v_8^2	v_8v_{12}	v_8v_{14}	v_8v_{15}	0	$v_8y_{11} + y_7v_{12}$	$v_8y_{13} + y_7v_{14}$.

Remark 3. Kuribayashi has also this result in [Ku].

Proof. We take $v_i = e^*(x_i),\ y_{i-1} = \tilde{\sigma}_{Id}(x_i) \quad (i = 8, 12, 14, 15)$. Just as in the previous calculations, we have

$$\begin{aligned}
&y_7^2 = Sq^7 y_7 = \tilde{\sigma}_{Id}(Sq^7 x_8) = \tilde{\sigma}_{Id}(x_{15}) = y_{14}\\
&Sq^1 y_i = \tilde{\sigma}_{Id}(Sq^1 x_{i+1}) = 0\ (i = 7, 11)\\
&Sq^1 y_{13} = \tilde{\sigma}_{Id}(Sq^1 x_{14}) = \tilde{\sigma}_{Id}(x_{15}) = y_{14} = y_7^2\\
&Sq^2 y_i = \tilde{\sigma}_{Id}(Sq^2 x_{i+1}) = 0\ (i = 7, 13)\\
&Sq^2 y_{11} = \tilde{\sigma}_{Id}(Sq^2 x_{12}) = \tilde{\sigma}_{Id}(x_{14}) = y_{13}\\
&Sq^4 y_i = \tilde{\sigma}_{Id}(Sq^4 x_{i+1}) = 0\ (i = 11, 13)\\
&Sq^4 y_7 = \tilde{\sigma}_{Id}(Sq^4 x_8) = \tilde{\sigma}_{Id}(x_{12}) = y_{11}\\
&Sq^8 y_7 = \tilde{\sigma}_{Id}(Sq^8 x_8) = 0\\
&Sq^8 y_{11} = \tilde{\sigma}_{Id}(Sq^8 x_{12}) = \tilde{\sigma}_{Id}(x_8 x_{12}) = y_7 v_{12} + v_8 y_{11}\\
&Sq^8 y_{13} = \tilde{\sigma}_{Id}(Sq^8 x_{14}) = \tilde{\sigma}_{Id}(x_8 x_{14}) = y_7 v_{14} + v_8 y_{13}\\
&y_{11}^2 = Sq^{11} y_{11} = \tilde{\sigma}_{Id}(Sq^{11} x_{12}) = \tilde{\sigma}_{Id}(Sq^1 Sq^2 Sq^8 v_{12}) = \tilde{\sigma}_{Id}(x_8 x_{15}) = v_8 y_7^2 + y_7 v_{15}\\
&y_{13}^2 = Sq^{13} y_{13} = \tilde{\sigma}_{Id}(Sq^{13} v_{14}) = \tilde{\sigma}_{Id}((Sq^5 Sq^8 + Sq^{11} Sq^2) x_{14})\\
&\qquad = \tilde{\sigma}_{Id}(Sq^5 x_8 x_{14}) = \tilde{\sigma}_{Id}(x_{12} x_{15}) = y_{11} v_{15} + v_{12} y_7^2\\
&y_7^4 = y_{14}^2 = Sq^{14} y_{14} = \tilde{\sigma}_{Id}(Sq^{14} x_{15}) = \tilde{\sigma}_{Id}(x_{14} x_{15}) = y_{13} v_{15} + v_{14} y_7^2.
\end{aligned}$$

□

Now we proceed to show that the mod 2 cohomology of $BSol(q) = L_{\psi^q} BDI(4)$ over $\mathcal{A}_2$ is isomorphic to that of $LBDI(4)$.

Proposition 5.2. *For $\phi = \psi^q$ the unstable Adams operation of degree an odd prime power q, $H^*(\mathbb{L}_\phi BDI(4); \mathbb{Z}/2)$ is isomorphic to $H^*(LBDI(4); \mathbb{Z}/2)$ as algebras over $\mathcal{A}_2$.*

Proof. Using the Wang sequence for $\mathbb{Z}/4$ and the Bockstein sequence, we can choose an element $u_8 \in \ker(Sq^1) \cap \ker(Sq^1 Sq^4) \subset (\iota^*)^{-1}(x_8)$ by the same observation in the proof of Proposition 3.2.

Put $u_{12} = Sq^4 u_8, u_{14} = Sq^2 u_{12}$ and $u_{15} = Sq^1 u_{14}$. Then we have $Sq^1 u_i = 0\ (i = 8, 12, 15)$. And by dimensional reason, we have $Sq^2 u_8 = 0$. Therefore $Sq^4 u_{12} = Sq^4 Sq^4 u_8 = (Sq^7 Sq^1 + Sq^6 Sq^2 + Sq^5 Sq^3) u_8 = 0$. Since $H^{19}(BDI(4); \mathbb{Z}/2) = 0$ and $\iota^*(Sq^8 u_{12}) = x_8 x_{12}$, using the Wang sequence for $\mathbb{Z}/2$ we have $Sq^8 u_{12} = u_8 u_{12}$. Other operations are calculated as follows:

$$\begin{aligned}
&Sq^2 u_{14} = Sq^2 Sq^2 u_{12} = 0\\
&Sq^4 u_{14} = Sq^4 Sq^6 u_8 = Sq^2 Sq^8 u_8 = 0\\
&Sq^8 u_{14} = Sq^8 Sq^2 u_{12} = (Sq^4 Sq^6 + Sq^2 Sq^8) u_{12} = u_8 u_{14}\\
&Sq^2 u_{15} = Sq^2 Sq^7 u_8 = Sq^9 u_8 = 0\\
&Sq^4 u_{15} = Sq^4 Sq^7 u_8 = Sq^{11} u_8 = 0\\
&Sq^8 u_{15} = Sq^8 Sq^1 u_{14} = (Sq^9 + Sq^2 Sq^7) u_{14} = Sq^1 Sq^8 u_{14} = u_8 u_{15}.
\end{aligned}$$

Hence we can construct a section r by $x_i \mapsto u_i$. □

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